CHARACTERISTICS OF THE FINAL FORM FOUND THROUGH THE REDUCED FORM: SYSTEMATIZING THE RELATIONSHIP BETWEEN THE THREE FORMS, STRUCTURAL, REDUCED, AND FINAL FORMS

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I Introduction

The equation of the final form of a model is expressed by

$$\sum_{j=0}^{pm} a_{ij} y_{it-j} = \sum_{j=1}^{q} \sum_{k=0}^{(p-1)m} b_{ijk} z_{jt-k} + \sum_{j=1}^{q} \sum_{k=0}^{(p-1)m} c_{ijk} e_{jt-k} \tag{1.1}$$

$$(i=1,\ 2,\ \cdots,\ p)$$

as shown by Klein,[1)] where y_{it-j} is the ith endogenous variable $(i=1, 2, \cdots, p)$ at time t-j, z_{it-k} the jth exogenous variable $(j=1, 2, \cdots, q)$ at time t-k $(k=0, 1, 2, \cdots, m)$, e_{jt-k} the jth residual at time t-k, and a_{ij}, b_{ijk} and c_{ijk} are parameters.

The purpose of this paper is (1) to show that we have a very simple and clear method to explain the form of the equation in the final form of a model, and (2) to systematize the relationship between the three forms of a model, structural, reduced, and final forms.

II The Final Form Obtained by Means of the Reduced Form

The model discussed here is complete and expressed by

$$\sum_{k=0}^{m} \boldsymbol{A}_k \boldsymbol{y}_{t-k} + \boldsymbol{B} \boldsymbol{z}_t = \boldsymbol{e}_t \tag{2.1}$$

where

$$\boldsymbol{A}_k = \begin{bmatrix} a_{11k} & a_{12k} \cdots a_{1pk} \\ \vdots & \vdots & \vdots \\ a_{p1k} & a_{p2k} \cdots a_{ppk} \end{bmatrix} \tag{2.2.1}$$

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \cdots b_{1q} \\ \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & b_{pq} \end{bmatrix} \tag{2.2.2}$$

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1) Klein, L. R.: *An Essay on the Theory of Economic Prediction.* Helsinki, Yrjö Jahnssonin Säätiö, 1968, pp. 15-17.
In Professor Klein's book written above, the process of derivation of the final form is stated based on the papar written by Dr. E. P. Howrey (Howrey, E. P.: *Dynamic Properties of Stochastic Linear Econometric Models.* Econometric Research Program, Memorandum, No. 87, Princeton University, June, 1967 (unpublished). According to the explanation, the equation (1.1) is written as follows:

$$\sum_{j=0}^{pm} a_{ij} y_{it-j} = \sum_{j=1}^{q} \sum_{k=1}^{(p-1)m} b_{ijk} z_{j-k} + \sum_{j=1}^{q} \sum_{k=1}^{(p-1)m} c_{ijk} e_{jt-k}$$

We can find minor differences between this equation and equation (1.1). We find "$k=0$" below the second Σ's of each term in the right side of equation (1.1), while in equation found in the explanation, we find "$k=1$" at the same places.

The present writer is afraid of his mistakes, and corrections by the reader are deeply appreciated.

$$\boldsymbol{y}_{t-k}=\begin{bmatrix} y_{1t-k} \\ y_{2t-k} \\ \vdots \\ y_{pt-k} \end{bmatrix} \tag{2.2.3}$$

$$\boldsymbol{z}_t=\begin{bmatrix} z_{1t} \\ z_{2t} \\ \vdots \\ z_{qt} \end{bmatrix} \tag{2.2.4}$$

$$\boldsymbol{e}_t=\begin{bmatrix} e_{1t} \\ e_{2t} \\ \vdots \\ e_{pt} \end{bmatrix} \tag{2.2.5}$$

and $\boldsymbol{A}_k$ and $\boldsymbol{B}$ are matrices of parameters, respectively.

As stated by Klein, if we use the lag operator L defined by

$$\boldsymbol{y}_{t-1}=L\boldsymbol{y}_t. \tag{2.3}$$

Then, the complete system expressed by equation (1.1) can be expressed by

$$\boldsymbol{A}(L)\boldsymbol{y}_t=-\boldsymbol{B}\boldsymbol{z}_t+\boldsymbol{e}_t \tag{2.4}$$

where

$$\boldsymbol{A}(L)=\boldsymbol{A}_0+\boldsymbol{A}_1L+\boldsymbol{A}_2L^2+\cdots+\boldsymbol{A}_mL^m \tag{2.5}$$

As the model is complete, we have $\boldsymbol{A}_0^{-1}$. Therefore, we can have the relation:

$$\boldsymbol{A}_0^{-1}\boldsymbol{A}(L)=\boldsymbol{E}+\boldsymbol{A}_1^*L+A_2^*L^2+\cdots+\boldsymbol{A}_m^*L^m \tag{2.6}$$

where

$$\boldsymbol{A}_k^*L^k=\boldsymbol{A}_0^{-1}\boldsymbol{A}_kL^k \quad (k=1,2,\cdots,m) \tag{2.7}$$

and $\boldsymbol{E}$ is the unit matrix.

And if we denote the $\boldsymbol{A}_0^{-1}\boldsymbol{A}(L)-\boldsymbol{E}$ by $\boldsymbol{A}^*(L)$, the equation (2.6) can be expressed by

$$\boldsymbol{A}^*(L)=\sum_{k=1}^{m}\boldsymbol{A}_k^*(L). \tag{2.8}$$

Moreover, we can express equation (2.4) by

$$\boldsymbol{y}_t=-\boldsymbol{A}^*(L)\boldsymbol{y}_t-\boldsymbol{B}^*\boldsymbol{z}_t+\boldsymbol{e}^*_t \tag{2.9}$$

where $\boldsymbol{B}^*$ is $\boldsymbol{A}_0^{-1}\boldsymbol{B}$ and $\boldsymbol{e}^*$ is $\boldsymbol{A}_0^{-1}\boldsymbol{e}_t$. This is the reduced form of the model questioned. The reduced form can be written by

$$\boldsymbol{S}(L)\boldsymbol{y}_t=-\boldsymbol{B}^*\boldsymbol{z}_t+\boldsymbol{e}^*_t \tag{2.10}$$

when we use the matrix $\boldsymbol{S}(L)$, where

$$\boldsymbol{S}(L)=\boldsymbol{E}+\boldsymbol{A}^*(L). \tag{2.11}$$

Then, the final form obtained from equation (2.10) is written by

$$|\boldsymbol{S}(L)|\boldsymbol{y}_t=-\boldsymbol{s}(L)\boldsymbol{B}^*\boldsymbol{z}_t+\boldsymbol{s}(L)\boldsymbol{e}^*_t \tag{2.12}$$

because we can obtain from equation (2.10),

$$(\boldsymbol{E}+\boldsymbol{A}^*(L))^{-1}(\boldsymbol{E}+\boldsymbol{A}^*(L))\boldsymbol{y}_t=-(\boldsymbol{E}+\boldsymbol{A}^*(L))^{-1}\boldsymbol{B}^*\boldsymbol{z}_t+(\boldsymbol{E}+\boldsymbol{A}^*(L))^{-1}\boldsymbol{e}^*_t \tag{2.13}$$

while the final form obtained from equation (2.4) is expressed by

$$|\boldsymbol{A}(L)|=-\boldsymbol{a}(L)\boldsymbol{B}\boldsymbol{z}_t+\boldsymbol{a}(L)\boldsymbol{e}_t \tag{2.14}$$

where $|\boldsymbol{S}(L)|$ and $|\boldsymbol{A}(L)|$ are the determinants of which elements are exactly equal to those of the matrices $\boldsymbol{S}(L)$ and $\boldsymbol{A}(L)$, respectively, and $\boldsymbol{s}(L)$ and $\boldsymbol{a}(L)$ are the cofactors of the matrices $\boldsymbol{S}(L)$ and $\boldsymbol{A}(L)$, respectively.

Firstly, we can say that $|\boldsymbol{S}(L)|$ is the polynomial of L^0, L^1, $\cdots$, L^m, L^{m+1}, $\cdots$, L^m, that is

$$|\boldsymbol{S}(L)|=\gamma_0L^0+\gamma_1L^1+\gamma_2L^2+\cdots+\gamma_{pm}L^{pm} \tag{2.15}$$

because the elements of $|\boldsymbol{S}(\mathrm{L})|$, $\gamma_{gh}(\mathrm{g}=1,2,\cdots,\mathrm{p})$ are the polynomial of L^0, L^1, $\cdots$, and L^m, where γ_u's $(u=1,2,\cdots,pm)$ are parameters.

Secondly, we can show that the elements of the cofactor $\boldsymbol{s}(L)$, $\Delta\sigma_{gh}$ is the polynomial of L^0, L^1, $\cdots$, L^m, L^{m+1}, $\cdots$, and $L^{(p-1)m}$ that is

$$\Delta\sigma_{gh}=\tau_{gh0}L^0+\tau_{gh1}L^1+\tau_{gh2}L^2+\cdots+\tau_{gh(p-1)m}L^{(p-1)m} \tag{2.16}$$

because the elements of the cofactors $\boldsymbol{s}(L)$, $\Delta\sigma_{gh}$ $(g=1, 2, \cdots, p;\ h=1,2,\cdots,p)$ are the determinants which have p-1 rows and p-1 columns and the elements of the determinants $\Delta\sigma_{gh}$, σ_{vw}'s $(v=1,2,\cdots,p\text{-}1,\ w=1,2,\cdots,p\text{-}1)$ are regarded as the functions of L^0, L^1, $\cdots$, and L^m, that is

$$\sigma_{vw}=t_{vw0}L^0+t_{vw1}L^1+t_{vw2}L^2+\cdots+t_{vwm}L^m \tag{2.17}$$

where $\tau_{ghk}(k=0,\ 1,\ 2,\ \cdots,\ (p\text{-}1)m)$ and $t_{vwk'}$ $(k'=1,\ 2,\ \cdots,m)$ are parameters, respectively, and especially, t_{vw0} is 0 or 1.

As shown above, we could show very clearly that the elements of cofactor $\boldsymbol{s}(L)$, $\Delta\sigma_{gh}$'s are the polynomials of L^0, $L^1,\cdots$, L^m, $\cdots$, $L^{(p-1)m}$ by using the reduced form of the model questioned.

III Systematizing the Three Forms of a Model

It was shown that the final form of a model can be obtained through the reduced form of the model, and we can know very easily and clearly the characteristics of the final form, by the final form obtained through the reduced form.

Here, we can systematize the three forms of a model.

If we write the structural form of a model by

$$\boldsymbol{A}_0\boldsymbol{y}_t+\boldsymbol{A}^L\boldsymbol{y}_t+\boldsymbol{B}\boldsymbol{z}_t=\boldsymbol{e}_t \tag{3.1}$$

we can show the reduced form of this model by

$$\boldsymbol{y}_t+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L\boldsymbol{y}_t+\boldsymbol{A}_0^{-1}\boldsymbol{B}\boldsymbol{z}_t=\boldsymbol{A}_0^{-1}\boldsymbol{e}_t \tag{3.2}$$

where

$$\boldsymbol{A}^L=\sum_{k=1}^{m}\boldsymbol{A}_kL^k. \tag{3.3}$$

Clearly, equation (3.2) is obtained by multiplying equation (3.1) by $\boldsymbol{A}_0^{-1}$. Equation (3.2) is written by

$$(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)\boldsymbol{y}_t+\boldsymbol{A}_0^{-1}\boldsymbol{B}\boldsymbol{z}_t=\boldsymbol{A}_0^{-1}\boldsymbol{e}_t. \tag{3.4}$$

Then, the final form can be expressed by

$$(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)^{-1}(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)\boldsymbol{y}_t+(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)^{-1}\boldsymbol{B}\boldsymbol{z}_t=(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)^{-1}\boldsymbol{A}_0^{-1}\boldsymbol{e}_t. \tag{3.5}$$

As shown by equation (3.5), the final form of a model is obtained by multiplying the structural form by two kinds of inverse matrices, $\boldsymbol{A}_0^{-1}$ and $(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^L)^{-1}$. If we call the number of the multiplications of a structural form by inverse matrices "degree of depth," or simply "depth," the structural, reduced and final forms can be called the form of which degrees of depth or depth are 0, 1, and 2, respectively.

And, in general, one of them can be called the form of which degree of depth is s $(s=0,\ 1,\ 2)$.

IV Examples of the Form of Which Degree of Depth Is s

Christ[2] showed a very interesting example of the final form of a model which was written by

$$C_t=\alpha Y_{t-1}+\beta+e \tag{4.1.1}$$

$$Y_t=C_t+I_t \tag{4.1.2}$$

where, C_t is consumption for the year t, Y_t income for the year t, and I_t investment for the year t, e residual.

The final form of this model is written by

$$C_t-\alpha C_{t-1}=\alpha I_{t-1}+\beta+e \tag{4.2.1}$$

$$Y_t-\alpha Y_{t-1}=I_t+\beta+e. \tag{4.2.2}$$

This is directly obtained from equations (4.1.1) and (4.1.2). The equations are expressed by

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}C_t\\ Y_t\end{bmatrix}+\begin{bmatrix}0L & -\alpha L\\ 0L & 0L\end{bmatrix}\begin{bmatrix}C_t\\ Y_t\end{bmatrix}+\begin{bmatrix}0 & -\beta\\ -1 & 0\end{bmatrix}\begin{bmatrix}I_t\\ Z_t\end{bmatrix}=\begin{bmatrix}e\\ 0\end{bmatrix} \tag{4.3}$$

or

$$\begin{bmatrix}1 & -\alpha L\\ -1 & 1\end{bmatrix}\begin{bmatrix}C_t\\ Y_t\end{bmatrix}+\begin{bmatrix}0 & -\beta\\ -1 & 0\end{bmatrix}\begin{bmatrix}I_t\\ Z_t\end{bmatrix}=\begin{bmatrix}e\\ 0\end{bmatrix} \tag{4.4}$$

where Z is a variable whose value is always 1. Equation (4.3) is the form of which degree of depth is 0 of the model. From equation (4.4), we can obtain the final form directly. The form is expressed by

2) Christ, Carl F. : *Econometrics*. NewYork, John Wily & Sons, 1960, pp. 176-186.

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} 1 & -\alpha L \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\beta \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix} = \begin{bmatrix} 1 & -\alpha L \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{4.5}$$

This equation is equivalent to the equation (4.2.1) and (4.2.2). On the other hand, we can obtain the reduced form or the form of which degree of depth is 1 from equation (4.3). The reduced form is written by

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0L & -\alpha L \\ 0L & 0L \end{bmatrix} \begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\beta \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{4.6}$$

This is the form of which degree of depth is 1. Incidentally, from this equation, we can obtain the following equations which have the usual form of the reduced form.

$$C_t = \alpha Y_{t-1} + \beta + e \tag{4.7.1}$$

$$Y_t = \alpha Y_{t-1} + I_t + \beta + e. \tag{4.7.2}$$

The reduced form expressed by equation (4.6) is obtained from equation (4.3) which can clearly express the structure of the equation with respect to the lagged variables. Therefore, it is adequate to call the equation (4.3) the form of which degree of depth is 0, although equations (4.3) and (4.4) are both the expressions of the structural form of the model.

We can get the final form or the form of which degree of depth is 2 from equation (4.6). The form of which degree of depth is 2 is written by

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0L & -\alpha L \\ 0L & 0L \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\beta \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0L & -\alpha L \\ 0L & \alpha L \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{4.8}$$

From this equation, we can obtain the following equations of the final form which are written in the usual form[3)]:

$$C_t - \alpha C_{t-1} = \alpha I_{t-1} + \beta Z + e \tag{4.9.1}$$

$$Y_t - \alpha Y_{t-1} = I_t + \beta Z + e. \tag{4.9.2}$$

Incidentally, according to the discussions by Christ, the solutions of C_t and Y_t are expressed by

$$C_t = C^\varepsilon + \alpha^t (C_0 - C^\varepsilon) \tag{4.10.1}$$

$$Y_t = Y^\varepsilon + \alpha^t (Y_0 - Y^\varepsilon) \tag{4.10.2}$$

where C^ε and Y^ε are the equillibrium values of C and Y, respectively.[4)]

In the model written above, we can find two exogenous variables I and β. One of them is a variate and the other is a constant.

Here, another example which is very simple is shown. The model in the following example was built by the present author.[5)] In this model, we have only one exogenous variable Z which is a constant. The model discussed here is written by

$$W_{Ut} = \alpha_0 Z + \alpha_1 D_{t-1} + e_1 \tag{4.11.1}$$

$$D_t = \beta_0 Z + \beta_1 W_{Ut} + e_2 \tag{4.11.2}$$

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3) Equation (5.8) becomes

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0L & -\alpha L \\ 0L & 0L \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & -\beta \\ -1 & -\beta \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0L & -\alpha L \\ 0L & 0L \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{1}$$

Therefore, we can have the relation:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} 1 & -\alpha L \\ 0 & 1-\alpha L \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\beta \\ -1 & -\beta \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix} = \begin{bmatrix} 1 & -\alpha L \\ 0 & 1-\alpha L \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{2}$$

From this equation, we get

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \frac{1}{1-\alpha L} \begin{bmatrix} 1-\alpha L & \alpha L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\beta \\ -1 & \beta \end{bmatrix} \begin{bmatrix} I_t \\ Z_t \end{bmatrix} = \frac{1}{1-\alpha L} \begin{bmatrix} 1-\alpha L & \alpha L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix}. \tag{3}$$

Therefore,

$$(1-\alpha L) \begin{bmatrix} C_t \\ Y_t \end{bmatrix} + \begin{bmatrix} -\alpha L I_t & -\beta Z_t \\ -I_t & -\beta Z_t \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}. \tag{4}$$

From this equation, we can get the relation:

$$(1-\alpha L) C_t - \alpha L I_t - \beta Z_t = e \tag{5.1}$$

$$(1-\alpha L) Y_t - I_t - \beta Z_t = e. \tag{5.2}$$

These equations are equivalent to equations (4.9.1) and (4.9.2).

4) Christ, Carl F.: op. cit., pp. 185–186.

5) Suzuki, Keisuke: *Keiryokeizaigakuteki Hoho no Kiso* (*Introduction to Econometric Methods*). Tokyo, Kotsu-Nihon-sha, 1976, pp. 261–270.

where W_t is the density of road in a region (which is the length of the road per one unit of area in a region) at time t, D_t the population density in a region at time t, e_1 and e_2 are residuals, α_0, α_1, β_0, and β_1 are parameters and Z is a constant whose value is 1. Therefore, the form of which degree of depth is 0 is expressed by

$$\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.12}$$

From this equation, we can get the reduced form or the form of which degree of depth is 1. The form is expressed by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.13}$$

And this equation is equivalent to the following equations.

$$W_{Ut}=\alpha_1 L D_t+\alpha_0 Z \tag{4.14.1}$$

$$D_t=\alpha_1\beta_1 L D_t+(\alpha_0\beta_1+\beta_0)Z+(\beta_1 e_1+e_2) \tag{4.14.2}$$

or

$$W_{Ut}=\alpha_1 D_{t-1}+\alpha_0 Z \tag{4.14$'$.1}$$

$$D_t=\alpha_1\beta_1 D_{t-1}+(\alpha_0\beta_1+\beta_0)Z+(\beta_1 e_1+e_2). \tag{4.14$'$.2}$$

The form of which degree of depth is 2 obtained from equation (4.18) is

$$\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z$$

$$=\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.15}$$

From this equation, we can get the following final form which has the usual expression.[6)]

$$W_{Ut}-\alpha_1\beta_1 W_{Ut-1}=\alpha_0+\alpha_1\beta_0+e_1+\alpha_1 e_2 \tag{4.16.1}$$

$$D_t-\alpha_1\beta_1 D_{t-1}=\alpha_0\beta_1+\beta_0+\beta_1 e_1+e_2. \tag{4.16.2}$$

On the other hand, we can obtain the final form directly from equation (4.12) as shown by equation (4.17).

$$\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}=\begin{bmatrix} 1 & -\alpha_1 L \\ -\beta & 1 \end{bmatrix}^{-1}\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}Z+\begin{bmatrix} 1 & -\alpha_1 L \\ -\beta_1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.17}$$

The equation (4.17) becomes

$$(1-\alpha_1\beta_1 L)\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}=\begin{bmatrix} 1 & \alpha_1 L \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}Z+\begin{bmatrix} 1 & \alpha_1 L \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{4.18}$$

or

$$W_{Ut}-\alpha_1\beta_1 W_{Ut-1}=\alpha_0+\alpha_1\beta_0+e_1+\alpha_1 e_2 \tag{4.18$'$.1}$$

$$D_t-\alpha_1\beta_1 D_{t-1}=\alpha_0\beta_1+\beta_0+\beta_1 e_1+e_2. \tag{4.18$'$.2}$$

These equations (equations (4.18), (4.18′.1), (4.18′.2)) are equivalent to equations (4.16.1) and (4.16.2).

The solutions of equation (4.16) or (4.18) are

$$W_{Ut}=\alpha_1\beta_1{}^t\left(W_{U0}-\frac{\alpha_0+\alpha_1\beta_0}{1-\alpha_1\beta_1}\right)+\frac{\alpha_0+\alpha_1\beta_0}{1-\alpha_1\beta_1} \tag{4.19.1}$$

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6) Equation (4.15) becomes

$$\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} 0L & -\alpha_1 L \\ 0L & 0L \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Therefore, (1)

$$\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} 1 & -\alpha_1 L \\ 0 & 1-\alpha_1\beta_1 L \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\begin{bmatrix} 1 & -\alpha_1 L \\ 0 & 1-\alpha_1\beta_1 L \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{2}$$

From this equation, we get

$$\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\frac{1}{1-\alpha_1\beta_1 L}\begin{bmatrix} 1-\alpha_1\beta_1 L & \alpha_1 L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\frac{1}{1-\alpha_1\beta_1 L}\begin{bmatrix} 1-\alpha_1\beta_1 L & \alpha_1 L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{3}$$

Therefore,

$$(1-\alpha_1\beta_1 L)\begin{bmatrix} W_{Ut} \\ D_t \end{bmatrix}+\begin{bmatrix} 1-\alpha_1\beta_1 L & \alpha_1 L \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_0 \\ -\beta_0 \end{bmatrix}Z=\begin{bmatrix} 1-\alpha_1\beta_1 L & \alpha_1 L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4}$$

From this equation, we can get the relation:

$$W_{Ut}-\alpha_1\beta_1 W_{Ut-1}-\alpha_0-\alpha_1\beta_0=e_1+\alpha_2 e_2 \tag{5.1}$$

$$D_t-\alpha_1\beta_1 D_{t-1}-\alpha_0\beta_1-\beta_0=\beta_1 e_1+e_2. \tag{5.2}$$

These equations are equivalent to equations (4.16.1) and (4.16.2).

$$D_t=\alpha_1\beta_1{}^t\left(D_0-\frac{\alpha_0\beta_1+\beta_0}{1-\alpha_1\beta_1}\right)+\frac{\alpha_0\beta_1+\beta_0}{1-\alpha_1\beta_1} \tag{4.19.2}$$

because, in general, the solution of the equation:

$$Y_t-aY_{t-1}=C \tag{4.20}$$

is

$$Y_t=a^t\left(Y_0-\frac{C}{1-a}\right)+\frac{C}{1-a} \qquad (4.21)^{7)}$$

where W_{U0}, D_0, and Y_0 are the values of W_U, D, and Y at time 0, respectively, Y_t is the value of a certain variable Y at time t, and C is a constant.

V Conclusion

We have three forms of a model, the structural, reduced, and final forms. The structure of the final form of a model is slightly complicated. But we can find that if we analyze the structure of the final form through the reduced form, the characteristics of the final form can be easily clarified.

And when we obtain the final form from the structural form through the reduced form, we can find a very interesting relationship between them.

Firstly, we can obtain the reduced form by multiplying the structural form by an inverse matrix. Secondly, we can also obtain the final form by multiplying the reduced form by another inverse matrix.

Then we can say that the structural form can be obtained by multiplying the structural form by no inverse matrix, the reduced form by multiplying of the structural form by one inverse matrix, and the final form by multiplying the structural form by two inverse matrices. Therefore, it can be said that the three forms of a model can be characterized by the number of multiplications of the structural form by the inverse matrices. If we call the number of multiplications of the structural form by the inverse matrices "the degree of depth" of a model, then the structural, reduced, and final forms can be called the forms of which degree of depth are 0, 1, and 2, respectively.

要　旨

鈴木啓祐『誘導型を通して見いだされる最終型の性質について：モデルの3形態，構造型，誘導型，および最終型の体系化』流通経済大学論集，第11巻，第2号，1976年10月，36—42頁。

完全体系の形で示されたモデルから得られる最終型の構造は，式(1.1)に示されるように，きわめて複雑である。しかしながら，この構造は，式(2.9)あるいは式(2.10)のように，はじめに，完全体系の誘導型を求め，これを基礎として式(2.12)のような形の最終型を求めると，最終型の構造が $\boldsymbol{S}(L)$ や $\boldsymbol{s}(L)$ の構造から比較的明確にとらえられる。ここでは，まず第1に，この点を明らかにした。

次いで，モデル，特に完全体系の形で示されたモデルがもつ3つの形態，すなわち，構造型(structural form)，誘導型(reduced form)，および，最終型(final form)が，それらを得るために用いられる逆行列の個数によって体系的に連関づけられることを示した。

クラインが示すように，完全体系をラグ・オペレーター(lag operator)を用いて，式(2.4)のように示し，この両辺に $\boldsymbol{A}(L)^{-1}$ を乗じることによって最終型を得ることができるが，このように，最終型を構造型から直接に求めず，式(4.2)に示すように，はじめ構造型から誘導型を求め，これを基礎として，式(4.5)のような形で最終型を求めることができる。

興味あることには，このような過程をへてモデルの各種の形態を得ると，つぎのようなことがいえる。まず，構造型は，構造型に逆行列を0回掛けることによって得られる。誘導型は，構造型に $\boldsymbol{A}_0^{-1}$ という逆行列を1回掛けることによって得られる。そして，最後に，最終型は，構造型に $\boldsymbol{A}_0^{-1}$ と $(\boldsymbol{E}+\boldsymbol{A}_0^{-1}\boldsymbol{A}^{L})^{-1}$ という逆行列をそれぞれ1回ずつ，合計2回掛けることによって得られる。いま，逆行列の掛ける回数を「深さ

7) Miyakawa, Kimio: *Keiryokeizaigaku* (*Econometrics*), Tokyo, Nihonkeizai Shimbunsha, 1966, pp. 86-88.

(degree of depth あるいは depth)」と名づければ，構造型，誘導型，および最終型は，それぞれ，「**深さ0，1および2の形態**」であるといえる。上記の3つの形態の相互関係を図示すれば，下図のようになるといえよう。